astro-ph/0404434 CGPG–04/4–5

Universality of the subsolar mass distribution from critical gravitational collapse

Matt Visser

School of Mathematical and Computing Sciences,
Victoria University of Wellington,
P.O. Box 600, Wellington, New Zealand

matt.visser@mcs.vuw.ac.nz

and

Nicolas Yunes

Center for Gravitational Physics and Geometry,
Center for Gravitational Wave Physics,
Department of Physics,
The Pennsylvania State University,
University Park, Pennsylvania 16802, USA

yunes@gravity.psu.edu

19 April 2004; LaTeX-ed February 2, 2008

ABSTRACT

Self-similarity induced by critical gravitational collapse is used as a paradigm to probe the mass distribution of subsolar objects. At large mass (solar mass and above) there is widespread agreement as to both the form and parameter values arising in the mass distribution of stellar objects. At subsolar mass there is still considerable disagreement as to the qualitative form of the mass distribution, let alone the specific parameter values characterizing that distribution.

For the first time, the paradigm of critical gravitational collapse is applied to several concrete astrophysical scenarios to derive robust qualitative features of the subsolar mass distribution. We further contrast these theoretically derived ideas with the observational situation. In particular, we demonstrate that at very low mass the distribution is given by a power law, with an exponent opposite in sign

to that observed in the high-mass regime. The value of this low-mass exponent is in principle calculable via dynamical systems theory applied to gravitational collapse. Qualitative agreement between theory, numerical experiments, and observational data is good, though quantitative issues remain troublesome.

Subject headings: Power laws, critical collapse, scaling, IMF; CGPG–04/4–5

1. Theoretical background

The initial mass function [IMF] describing the mass distribution of stellar objects is one of the basic tools used in studying the evolution and lifecycles of stars and galaxies. At large mass, the IMF is well-characterized by a power law with the Salpeter exponent 1.35. At small sub-solar mass the situation is much more uncertain. Using theoretical ideas based on self-similarity and critical gravitational collapse, we argue in a model-independent manner that there *must* be a change in this power law, and in fact that the sign of the exponent must flip. Contrasting this with direct observation indicates that the IMF is certainly modified below approximately 0.8 $M_\odot$, and the theoretical expectation of a sign flip is borne out by the data. [The current article fleshes out some of the details backing up the comments briefly summarized in Visser and Yunes (2004).]

Our key observation starts from the well-known fact that gravitational condensation, either Newtonian or general relativistic, is characterized by the existence of critical exponents and power-law behavior. By linearizing around any critical solution at the threshold of collapse, the mass M of the resulting condensed object is related to any suitable control parameter A in the initial data by an equation of the form (Harada *et al.* 2003; Maeda and Harada 2001; Gundlach 1998a,b, 1999, 2003)

$$M \approx M_0 \left[A - A_{\text{critical}}\right]^{\delta}; \qquad \delta > 0. \tag{1}$$

The fact that scaling of this type often occurs is *generic*, and independent of the details of the collapse process and equation of state — and independent of whether or not the collapse is relativistic. The precise *value* of the scaling exponent does depend on the specific details of the collapse process, but the fact that such scaling occurs is intrinsically universal.

Once a scaling law of this type is derived, we shall see that straightforward manipulations lead to a power law for the distribution of low-mass objects

$$P(M) \approx \frac{\mathcal{A}}{M_0} \left(\frac{M}{M_0}\right)^{(1/\delta)-1}, \tag{2}$$

with an exponent that is calculable in terms of the mass-scaling exponent, δ. In this manner, we can explain the low-mass tail in the Initial Mass Function [IMF] from first principles in terms of dynamical systems theory in gravitational collapse. The technique developed in this article cannot say anything about the high-mass tail of the IMF, but that is a regime where there is reasonable theoretical and observational agreement on the state of affairs. We shall specifically concentrate on the functional form of the IMF for subsolar masses.

To set the stage, recall that any gravitationally self-interacting cloud of gas, either Newtonian or general relativistic, has a limited number of long-term fates:

- The cloud can completely disperse to infinity.
- Part of the cloud might condense, with the remainder dispersing to infinity.
- The entire cloud might condense.

The condensed object could, for instance, be a solid planet, a fluid star, or a black hole, although it does not need to collapse to infinite density. The set of all initial data that lead to any one of these fates can be thought of as an infinite-dimensional phase space, containing infinite-dimensional basins of attraction for each final fate. Since there are three possible final fates for a cloud of gas, there will be three basins of attraction: the collapse basin, where its attractor leads to complete collapse; the dispersal basin, for which the final fate is an asymptotically flat Minkowski spacetime; and an intermediate collapse basin, where ultimately part of the cloud collapses and the rest disperses to infinity. These basins will be separated from each other by boundaries of co-dimension one, or separatrices, that form the so-called critical surfaces. In this manner, it is clear that the critical surfaces contain all critical initial data that separate two basins of attraction. An example of critical initial data, *i.e.* a point on the critical surface, would be the Jeans mass, or Jeans energy. Another important point on this surface will be an intermediate attractor in phase space, which will be referred to as the critical solution. This critical solution will have important properties, such as self-similarity or scale-invariance. For a more complete and detailed analysis refer to (Gundlach 1998a,b, 1999, 2003).

Applying dynamical systems theory to a region of phase space close to any collapse-dispersal separatrix leads generically to the prediction of power-law behavior for the mass of the resulting condensed object. In order to make this point more explicit, let us consider some set of initial data parameterized by the control parameter A. Let us also assume that for $A < A_{\text{critical}}$ the cloud completely disperses, while for $A > A_{\text{critical}}$ at least part of the cloud condenses. In other words, if A lies inside of the intermediate collapse basin, the solution to the field equations will be equivalent to finding an integral curve in phase space

from A to the final attractor of this basin. Similarly, if A lies inside of the dispersal basin, then the integral curve will start at A but end at the final attractor of dispersal. Then, under the mild technical assumption of the existence of at least one critical collapse solution on the critical surface with exactly one unstable mode (Harada *et al.* 2003; Maeda and Harada 2001; Gundlach 1998a,b, 1999, 2003), the condensed mass will be given by

$$M \approx M_0 \left[A - A_{\text{critical}}\right]^{\delta}, \tag{3}$$

provided that the initial data is chosen reasonably close to the critical surface, *i.e.* $A \approx A_{\text{critical}}$.

The physically interesting quantity is the exponent δ, which arises naturally as the fractional power-series exponent of a generalized Frobenius expansion for linear perturbations around the critical solution (Visser and Yunes 2003). The order parameter A, the value of the critical initial data A_{critical}, and the constant of proportionality M_0 can be changed at will by reparameterizing the initial data set. In contrast, the exponent δ is physically significant and reparameterization invariant, depending only on the equation of state and the condensation mechanism. Observe that, by construction, we must have $\delta > 0$, since $\delta < 0$ would imply an abrupt transition between no condensation and complete condensation of the cloud. Even $\delta = 0$ is problematic, since this corresponds to an abrupt transition from no condensation to a finite condensate mass. It is only for $\delta > 0$ that as we fine-tune the control parameter A we get the physically reasonable situation of no condensation connected smoothly to a low mass condensate for $A > A_{\text{critical}}$.

Behavior of this type has by now been seen in numerical experiments in a considerable number of seemingly disparate situations.

- In Newtonian gravity coupled to a gas cloud with some specified equation of state, such as an isothermal one, it is possible to observe the same scaling behavior of the mass. First, one searches for solutions describing critical collapse and then linearizes around these critical collapse solutions to find δ (Harada *et al.* 2003; Maeda and Harada 2001). The Newtonian isothermal collapse case is of particular relevance in astrophysics, since it is a good description for cold molecular gas in the interstellar medium, where the cooling time is much shorter than the dynamical time.

- In general relativity, the special case where the condensed object is a black hole is known as Choptuik scaling (Gundlach 1998a,b, 1999, 2003; Choptuik 1993; Choptuik *et al.* 1996). This phenomenon has now been analyzed not just for gas clouds but also for several other forms of matter. In particular, the analyses in Choptuik (1998) and Koike *at al.* (1999) showed that for an adiabatic perfect fluid with adiabatic index in

the domain $\gamma \in (1, 1.89)$, where $p = (\gamma - 1)\rho c^2$, the critical exponent varies over the range $\delta \in (0.106, 0.817)$, clearly demonstrating the dependence of this exponent on the equation of state.

Several key results are summarized in Table I.

Critical exponents determined by numerical experiment.

System	Critical Point	Exponent δ	$1/\delta$
Newtonian isothermal	Hunter A	0.10567	9.4637
GR dust: $p = 0$	Evans–Coleman	0.10567	9.4637
GR radiation: $p = \frac{1}{3}\rho c^2$	Evans–Coleman	0.3558019	2.810553
GR semi-stiff: $p = \frac{4}{5}\rho c^2$	Evans–Coleman	0.73	1.37
GR stiff: $p = \rho c^2$	Evans–Coleman	0.96	1.04

Table I: Key known values of critical exponents in various systems.
See references (Harada *et al.* 2003; Maeda and Harada 2001; Gundlach 1998a,b, 1999, 2003; Choptuik 1998; Koike *at al.* 1999) and references therein.

Although the phenomenon of Choptuik scaling is concerned with black hole formation, the qualitative features of the collapse mechanism are identical to that of stellar formation. Matter subjected to gravity will generically condense, at least partially, irrespective of whether it is Einstein's or Newton's gravity that rules. Furthermore, this condensation will generically occur regardless of the initial form or distribution of matter. The collapsing matter knows nothing about its final fate, or which basin of attraction it lies in. Hence, the presence or absence of critical mass scaling will be independent of the final fate of the system.

In this manner, it is clear that the details of the separatrix that is being considered are irrelevant, as far as the existence and numerical value of mass scaling critical exponents are concerned. The precise values of these exponents will depend on the collapse mechanism and equation of state, and not on the final fate of the cloud. When we confront our theoretical ideas with observation, we will use the numerical exponents found for black hole formation [see (Harada *et al.* 2003; Maeda and Harada 2001; Gundlach 1998a,b, 1999, 2003; Choptuik 1998; Koike *at al.* 1999) and references therein], to attempt to pin down the nature of the specific critical collapse process that actually leads to the formation of subsolar stellar objects.

2. From critical collapse to IMF

Extending this analysis further, suppose a number of Newtonian or relativistic systems, with initial data depending on some control parameter A, evolve dynamically. Let the distribution of initial control parameters be given by the probability distribution function $P_a(A)$. We can then determine the probability $P(M) \propto \mathrm{d}N/\mathrm{d}M$ of producing low-mass condensed objects by calculating

$$P(M)\,\mathrm{d}M = P_a(A)\,\frac{\mathrm{d}A}{\mathrm{d}M}\,\mathrm{d}M. \tag{4}$$

We can invert equation (3), to obtain

$$A(M) \approx A_{\mathrm{critical}} + \left(\frac{M}{M_0}\right)^{1/\delta}, \tag{5}$$

and so rewrite the probability as

$$P(M)\,\mathrm{d}M \approx \frac{1}{\delta}\,\frac{P_a(A)}{M_0}\,\left(\frac{M}{M_0}\right)^{(1/\delta)-1}\,\mathrm{d}M. \tag{6}$$

This can further be written as

$$P(M)\,\mathrm{d}M \approx \frac{1}{\delta}\,\frac{P_a(A_{\mathrm{critical}} + [M/M_0]^{1/\delta})}{M_0}\,\left(\frac{M}{M_0}\right)^{(1/\delta)-1}\,\mathrm{d}M. \tag{7}$$

As long as $P_a(A)$ is smooth enough to have a Taylor series expansion we can write

$$P(M)\,\mathrm{d}M \approx \frac{1}{\delta}\,\left\{P_a(A_{\mathrm{critical}})\,\left(\frac{M}{M_0}\right)^{(1/\delta)-1} + \left.\frac{\mathrm{d}P_a}{\mathrm{d}A}\right|_{A_{\mathrm{critical}}}\left(\frac{M}{M_0}\right)^{(2/\delta)-1}\right\}\,\frac{\mathrm{d}M}{M_0}. \tag{8}$$

As long as δ is positive, there will be some region for which the linear term and all higher-order terms can safely be neglected, since M is assumed small as compared to M_0. We can then rewrite the probability as

$$P(M)\,\mathrm{d}M \approx \frac{1}{\delta}\,P_a(A_{\mathrm{critical}})\,\left(\frac{M}{M_0}\right)^{(1/\delta)-1}\,\frac{\mathrm{d}M}{M_0}, \qquad \text{as} \quad M \ll M_0. \tag{9}$$

This restriction that $P_a(A)$ have a well-behaved Taylor expansion near the critical value of the order parameter is not strong at all. With this in mind, we expect for low mass objects a power law distribution in masses:

$$P(M \ll M_0) \approx \frac{\mathcal{A}}{M_0}\,\left(\frac{M}{M_0}\right)^{(1/\delta)-1} = \frac{\mathcal{A}}{M_0}\,\left(\frac{M}{M_0}\right)^{n-1}, \tag{10}$$

where we have defined $n \equiv 1/\delta$. We must emphasize that the previous equation is completely equivalent to either of the following forms:

$$P(M \ll M_0) \approx \frac{1}{M_0} \left(\frac{M}{M_\delta}\right)^{(1/\delta)-1} = \frac{1}{M_0} \left(\frac{M}{M_\delta}\right)^{n-1}. \tag{11}$$

or

$$P(M \ll M_0) \approx \frac{1}{\widetilde{M}_\delta} \left(\frac{M}{\widetilde{M}_\delta}\right)^{(1/\delta)-1} = \frac{1}{\widetilde{M}_\delta} \left(\frac{M}{\widetilde{M}_\delta}\right)^{n-1}. \tag{12}$$

Any of these three forms differ from each other only by a redefinition of the parameters appearing in the power law. This implies an extreme flexibility in the appearance of the power law, which may sometimes disguise the equality of two apparently different presentations. The only true invariant of a power law is the exponent δ (or n). The mass parameters M_0, M_δ, and $\widetilde{M}_\delta$, can be redefined almost at will.

Furthermore, observe that this entire analysis holds only for small masses, since we have assumed that the control parameter A is near the critical surface. This behavior is structurally similar to the observed high-mass IMF, $\xi(M)$, given by

$$\xi(M) = \frac{\mathrm{d}N}{\mathrm{d}M} = N_{\text{total}}\, P(M), \tag{13}$$

where N_{total} is the total number of stars in the region of interest and at large mass the probability function is a power law of the form

$$P(M \gg M_0) \approx \frac{\mathcal{B}}{M_0} \left(\frac{M}{M_0}\right)^{-m-1}. \tag{14}$$

At large mass observation favors the Salpeter exponent $m \approx 1.35$. [Unfortunately not all authors agree on the precise definition of $\xi(M)$, see Miller and Scalo (1979) for a brief discussion, and accordingly some care must be taken in comparing IMFs extracted from different sources.] We choose to work directly with the probability distribution $P(M)$ normalized so that

$$\int_0^\infty P(M)\, \mathrm{d}M = 1. \tag{15}$$

The major difference at low mass is that the sign of the exponent changes, which is necessary on two counts: in order that the probability function be integrable, and that the exponent δ be even in principle calculable within the current scenario.

3. Modeling the IMF

The simplest toy model that exhibits both forms of asymptotic behavior and that is similar to models widely used by astronomers to describe the observations is

$$P(M) = \frac{n\, m}{n+m}\, \frac{1}{M_0} \left\{ \left(\frac{M}{M_0}\right)^{+n-1} \Theta(M_0 - M) + \left(\frac{M}{M_0}\right)^{-m-1} \Theta(M - M_0) \right\}, \tag{16}$$

where both n and m are positive. This toy model provides us with a well-behaved normalizable probability distribution

$$\int P(M)\, \mathrm{d}M = 1; \qquad (m > 0;\ \ n > 0); \tag{17}$$

with finite mean and variance:

$$\overline{M} = \int M\, P(M)\, \mathrm{d}M = \frac{nm}{(n+1)(m-1)} M_0; \qquad (m > 1;\ \ n > 0) \tag{18}$$

$$\sigma^2 = \int (M - \overline{M})^2\, P(M)\, \mathrm{d}M = \frac{nm[(m-1)^2 + (n-1)^2 - 1]}{(n+2)(n+1)^2(m-2)(m-1)^2}\, M_0^2; \qquad (m > 2;\ \ n > 0). \tag{19}$$

This is the simplest model that is realistic in terms of being well behaved at both the upper and lower limits.

More generally, one might wish to consider piecewise power laws as suggested by Miller and Scalo (1979); Scalo (1986); Kroupa (2001). Note that interest in these piecewise power laws is dictated by their mathematical and observational convenience — there is no sound physical motivation for the abrupt change in behaviour encoded in the Heaviside function. However, this distribution proves effective in summarizing the observational data with a minimum of complications. A generalized representation of these piecewise power laws is

$$P(M) = \sum_{i=0}^{N} \frac{\mathcal{B}_i}{\mu}\ \ \theta(M - M_i)\ \ \theta(M_{i+1} - M)\ \ \left(\frac{M}{\mu}\right)^{n_i - 1}, \tag{20}$$

with $M_0 = 0$, and $M_N = \infty$. Here μ is any conveniently chosen arbitrary but fixed mass scale, and the coefficients $\mathcal{B}_i$ are chosen so as to make $P(M)$ continuous:

$$\mathcal{B}_i \left(\frac{M_{i+1}}{\mu}\right)^{n_i - 1} = \mathcal{B}_{i+1} \left(\frac{M_{i+1}}{\mu}\right)^{n_{i+1} - 1} \tag{21}$$

That is

$$\mathcal{B}_i = \mathcal{B}_{i+1} \left(\frac{M_{i+1}}{\mu}\right)^{n_{i+1} - n_i} \tag{22}$$

Observe that although this probability function is continuous, it is not differentiable at the interfaces $M = M_i$, leading to "kinks". This non-differentiability is due to the fact that there is only one degree of freedom available in the $\mathcal{B}_i$ and it has been already used to impose continuity.

In order for this piecewise power law distribution to be normalizable we must demand $n_0 > 0$, and $n_N < 0$. To additionally obtain a finite mean we need $n_0 > 0$, and $n_N < -1$. For a finite variance we need $n_0 > 0$ and $n_N < -2$. Thus the values of both the low-mass n_0 exponent and high mass n_N exponent have important implications for the mathematical existence of a normalizable probability distribution, with finite mean and standard deviation. We shall confront these theoretical considerations with the observational data in a later section.

First, however, let us consider the possibility that there are several *different* and *independent* competing collapse processes with different critical solutions, indexed by the label α. Then, for *each* of these processes we will have an independent mass scaling law of the form

$$M \approx M_{\delta_\alpha} \, (A - A_{\text{critical}})^{\delta_\alpha}. \tag{23}$$

If the condensation is known to take place via process α, then this leads to a probability distribution given by

$$P_\alpha(M) \approx \frac{\mathcal{A}_\alpha}{M_{\delta_\alpha}} \left(\frac{M}{M_{\delta_\alpha}}\right)^{(1/\delta_\alpha)-1}. \tag{24}$$

Since none of these competing processes depend on each other, each will contribute independently with probability p_α to the total probability distribution, leading to

$$P_{\text{total}}(M) \approx \sum_\alpha p_\alpha \, P_\alpha(M) = \sum_\alpha p_\alpha \, \frac{\mathcal{A}_\alpha}{M_{\delta_\alpha}} \left(\frac{M}{M_{\delta_\alpha}}\right)^{(1/\delta_\alpha)-1}. \tag{25}$$

By redefining parameters we can rewrite this to provide an alternative but equivalent representation of the form

$$P_{\text{total}}(M) \approx \sum_\alpha \frac{\mathcal{B}_\alpha}{\mu} \left(\frac{M}{\mu}\right)^{(1/\delta_\alpha)-1} = \sum_\alpha \frac{\mathcal{B}_\alpha}{\mu} \left(\frac{M}{\mu}\right)^{n_\alpha - 1}. \tag{26}$$

We emphasize that this is a sum over *distinct* and *independent* critical solutions, and so is not intrinsically a "piecewise power law" of the type considered above. Although this physically motivated distribution possesses the same general shape as the "kinked" probability law aforementioned, it smoothes out the "kinks", allowing for differentiability at the interfaces. The general shape of the "kinked" power law is retained because, the largest of the δ_α will dominate at the smallest masses. Eventually, there will be an approximate switch-over to

one of the other critical exponents at larger mass. If this larger mass is still reasonably small, one could calculate it using dynamical system theory. In this manner, one may hope to approximately model the observationally-motivated piecewise power-law IMF all the way up to its peak with a physically-motivated differentiable power-law. However, one can never obtain the high-mass decreasing tail from this sort of analysis.

Finally, we should mention the possibility of using Gamma and inverse-Gamma distributions as building blocks for the IMF probability function. These distributions require additional physical input in the form of an exponential cutoff, which, although not well motivated physically, allows for the construction of global fits to the data. For the Gamma distribution

$$P_n(M) = \frac{1}{\Gamma(n)} \left(\frac{M}{M_0}\right)^{n-1} \exp\left\{-\frac{M}{M_0}\right\}. \tag{27}$$

The probability integral converges for $n > 0$, and has finite mean and variance

$$\overline{M} = \int M \; P(M) \; \mathrm{d}M = n \; M_0; \tag{28}$$

$$\sigma^2 = \int (M - \overline{M})^2 \; P(M) \; \mathrm{d}M = n \; M_0^2. \tag{29}$$

This Gamma distribution is most useful for low mass where it provides an accurate approximation to a pure power law, while at high mass the exponential cutoff keeps everything finite. In contrast, the inverse-Gamma distribution is

$$P_m(M) = \frac{1}{\Gamma(m)} \left(\frac{M}{M_0}\right)^{-m-1} \exp\left\{-\frac{M_0}{M}\right\}. \tag{30}$$

The probability integral now converges for $m > 0$, and has finite mean and variance

$$\overline{M} = \int M \; P(M) \; \mathrm{d}M = \frac{M_0}{m-1}; \qquad (m > 1) \tag{31}$$

$$\sigma^2 = \int (M - \overline{M})^2 \; P(M) \; \mathrm{d}M = \frac{M_0^2}{(m-2)(m-1)^2}; \qquad (m > 2). \tag{32}$$

The inverse-Gamma distribution is most useful for high mass where it provides an accurate approximation to a pure power law, while at low mass the exponential cutoff keeps the total probability finite.

Arbitrarily complicated probability distributions could now be constructed by taking linear combinations of Gamma and inverse-Gamma distributions. This is, however, not the way things have historically been done in the observational literature. These comments on Gamma distributions are mentioned here as a potentially useful representation for future study in observational astrophysics.

4. Observational situation

In contrast to these theoretical considerations, direct astrophysical observation leads to several models for $P(M)$ that are piecewise power laws (Table II), and to several isolated data points at low mass (Table III). The three standard IMFs are those of Salpeter (Salpeter 1955), Miller–Scalo (Miller and Scalo 1979), and Scalo (Scalo 1986), with a more recent version due to Kroupa (Kroupa 2001). Relatively few of the ranges in Table II correspond to a positive δ. For low mass condensates, Scalo gives $m = -1/\delta = -2.60$ so that $\delta = 0.385$, while Kroupa gives $m = -1/\delta \in (-1.4, 0.0)$ so that $\delta \in (0.71, \infty)$. All the other parts of the standard IMFs correspond to the high mass region where the number density is decreasing with increasing mass.

Multi-scale observational IMFs.

IMF: $P(M) = (\mathcal{A}/M_0)\ (M/M_0)^{-m-1}$	$M_1/M_\odot$	$M_2/M_\odot$	Exponent m
Salpeter (1955)	0.10	125	1.35
Miller and Scalo (1979)	0.10	1.00	0.25
	1.00	2.00	1.00
	2.00	10.0	1.30
	10.0	125	2.30
Scalo (1986)	0.10	0.18	-2.60
	0.18	0.42	0.01
	0.42	0.62	1.75
	0.62	1.18	1.08
	1.18	3.50	2.50
	3.50	125	1.63
Kroupa (2001)	0.01	0.08	-0.7 ± 0.7
	0.08	0.50	$+0.3 \pm 0.5$
	0.50	∞	1.3 ± 0.3

Table II: Observationally derived piecewise power-law $P(M)$.

Those IMFs obtained using observations which focused on the substellar regime are summarized in Table III. These observations indicate broad observational agreement as to the sign of the low-mass exponent, and a preponderance of evidence pointing to a clustering of the exponent at $m \approx -0.5$, *i.e.* $n \approx +0.5$ and $\delta \approx +2$. These low-mass exponents are converted into critical exponents in Table IV. By comparing the theoretical results in Table I with the observational results in Table IV, we can see that while there is broad agreement

between observation and theory regarding the sign of the exponent, *quantitative* agreement is more problematic.

Low-mass observational IMF.

IMF: $P(M) = (\mathcal{A}/M_0)\ (M/M_0)^{-m-1}$	$M_1/M_\odot$	$M_2/M_\odot$	Exponent m
Barrado y Navascues *et al.* (2000)	0.2	0.8	-0.2
Barrado y Navascues *et al.* (2002)	0.035	0.3	-0.4
Bouvier *at al.* (1998, 2003)	0.03	0.48	-0.4
Martin *at al.* (2000)	0.02	0.1	-0.47
Bouvier *at al.* (2002)	0.072	0.4	-0.5
Luhman (1999)	0.02	0.1	-0.5
Najita *et al.* (2000)	0.015	0.7	-0.5
Rice at al. (2003)	10^{-5}	10^{-3}	≈ -1
Tej et al. (2002)	0.01	0.50	-0.2 ± 0.2
Tej et al. (2002)	0.01	0.50	-0.5 ± 0.2

Table III: Observationally derived low-mass $P(M)$.

A particularly nice feature is that the observationally derived low mass exponent $n_0 = -m_0 \approx 1/2$ is compatible with a normalizable probability distribution. The observationally determined high mass exponent (the Salpeter exponent) $m_N = -n_M \approx 1.35$ is compatible with a normalizable probability distribution of finite mean, but with an infinite variance arising from the high-mass tail. Probability distributions of finite mean but infinite variance are well-known in statistics, and while they make perfectly good sense mathematically they are associated with perhaps unexpected mathematical subtleties (such as, for instance, the failure of the central limit theorem). Note that this behavior is coming from the high-mass region, not the low-mass region that is of primary concern in the current article.

We must conclude that present day observational data is sufficiently poor that the only rigorous inference one can draw is that the exponent has changed sign at sufficiently low masses. Beyond that, it would be desirable to contrast the exponent occurring in the subsolar IMF with the exponent arising in a specific critical collapse process. Unfortunately, neither observational data nor theory is currently well enough developed to do so with any degree of reliability.

Some of the numerical simulations give critical exponents that overlap with some of the observations. For instance, the Scalo exponent is roughly comparable with that arising from numerical simulations of collapse of a relativistic radiation fluid, $p = \frac{1}{3}\rho c^2$. Part of the

Observed low-mass exponents.

Source	Exponent m	Exponent $1/\delta$	Exponent δ
Scalo (1986)	-2.60	2.60	0.385
Kroupa (2001)	-1.4 — 0.0	0.0 — 1.4	0.71 — ∞
Rice at al. (2003)	≈ -1	≈ 1	≈ 1
Najita *et al.* (2000)	-0.5	0.5	2.0
Luhman (1999)	-0.5	0.5	2.0
Bouvier *at al.* (2002)	-0.5	0.5	2.0
Martin *at al.* (2000)	-0.47	0.47	2.16
Bouvier *at al.* (1998, 2003)	-0.4	0.4	2.5
Barrado y Navascues *et al.* (2002)	-0.4	0.4	2.5
Barrado y Navascues *et al.* (2000)	-0.2	0.2	5.0
Tej et al. (2002)	-0.5	0.5	2.0
Tej et al. (2002)	-0.2	0.2	5.0

Table IV: Observational estimates of the very low mass exponents.

range of Kroupa's IMF, *i.e.* $\delta \in (0.71, 1)$, is compatible with simulations of a relativistic adiabatic perfect fluid, $p = k\ \rho c^2$ with $k \in (\frac{4}{5}, 1)$. Finally, the IMF exponent of Rice *et al* is compatible with a numerical critical solution corresponding to a relativistic stiff fluid, $p = \rho c^2$. Those observations that cluster around $\delta = 2$ are not compatible with any *known* critical collapse solution. This might indicate either a problem with the observational data, or a more fundamental lack of understanding regarding the physically relevant critical collapse process.

5. Conclusions

Future work along these lines should be focused in two directions. Observationally, improved data would be desirable to test the hypothesis that the low-mass exponent δ is both positive and universal. Theoretically, it would be important to understand *quantitatively* why critical behavior provides an accurate representation of the IMF for $M \lesssim 0.8\ M_{\odot}$. It is clear that as the final condensed mass increases, the initial data A is pushed farther away from the critical surface, *i.e.* $A \neq A_{\text{critical}}$. Although it is known that the linear perturbation around the critical solution then loses validity, a precise calculation of the region of convergence is still lacking.

The formation of real-world gravitational condensates is likely to involve rotating turbulent dust clouds. Therefore, it would be very useful to understand the influence of both angular momentum and turbulence on the theoretically derived critical exponents. Physical intuition suggests that turbulence would make the collapse process even more scale-invariant, since it would eliminate all irregularities in the collapse and favor self-similar behavior. In this manner, the scaling law and critical exponent presented for spherical collapse should not be modified by turbulence, but instead it should be made more predominant.

A detailed analysis of angular momentum presents new challenges to the critical behavior framework by sometimes introducing a second non-spherical growing mode that competes with the usual growing mode (Gundlach 2002). The presence or absence of a second growing mode depends on the equation of state, and is known to occur for some specific polytropic equations of state. These additional growing modes possess different eigenvalues from the spherical one, leading to new scaling exponents. In particular, angular momentum will scale with a new critical exponent and will possess a new critical parameter in the initial data. In this manner, the probability distribution becomes now two-dimensional, depending both on mass and angular momentum.

At the present time, it is not clear precisely how the functional form of the mass scaling law will be modified by the inclusion of angular momentum. For main sequence stars, however, physical intuition suggests that low-angular momentum should be dominant. Hence, corrections to the mass scaling law based on a spherically symmetric idealization should be small, though not necessarily negligible.

Our analysis confirms Larson's intuition that stellar formation at low mass is related (and perhaps even dominated) by chaotic dynamics (Larson 2002). In particular, the analysis in terms of dynamical systems theory can be viewed in terms of deterministic chaos in gravitational collapse. We do not, however, need to deal with fractal structures since limit points and limit cycles seem to be quite sufficient for generating power-law behavior (Visser and Yunes 2003). Our analysis further supports the idea of a universal slope, dependent only on the relevant critical collapse solution, but independent of the initial conditions, and disfavors the astrophysical hypothesis of an IMF that varies in both time and space.

Summarizing, the dynamical exponents found in Newtonian and general relativistic gravitational collapse can be used to model and qualitatively explain a power law version of the IMF valid for small masses. For the first time, a concrete application to the numerical phenomena of critical gravitational collapse has been proposed and tested against observational data. We have compared these results to subsolar IMF data and found them in broad qualitative agreement for low-mass systems, though quantitative agreement is poor at this stage. The key point is that gravitational collapse naturally leads to power law behavior in

the low mass regime, with an exponent that is opposite in sign to the observed high-mass behavior. This provides a new and fresh view on power-law behavior with specific astrophysical applications to dynamic gravitational collapse and the IMF.

Acknowledgments: We are grateful to Ben Owen and Tom Abel for insightful comments on gravitational collapse theory and astrophysical observations of the IMF. We would also like to acknowledge Carsten Gundlach for clarifying some issues regarding the dynamical systems framework.

Nicolas Yunes wishes to acknowledge the support of the Center for Gravitational Physics and Geometry, the Center for Gravitational Wave Physics and the Physics Department at The Pennsylvania State University.

The research of Nicolas Yunes was supported partly by the US NSF grants PHY-0114375 and PHY-0245649, while the research of Matt Visser was supported by a Marsden Grant administered by the Royal Society of New Zealand.

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